J. Fluid Mech. (1975), *vol.* 69, *part* 1, *pp.* 65–72
Printed in Great Britain

The 'Richardson' criterion for compressible swirling flows

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(Received 2 May 1974)

The stability of a compressible non-dissipative swirling flow to adiabatic infinitesimal disturbances of arbitrary orientation is considered. The resulting sufficient condition for stability is the general form of the effective Richardson criterion for swirling flows, first obtained, for axisymmetric modes only, by Howard. In addition, upper bounds to the growth rate of unstable modes are obtained and some extensions of the semicircle theorem to azimuthal disturbances are stated.

1. Introduction

Many of the results on the stability of variable density, inviscid shear flows in the presence of gravity (Miles 1961; Howard 1961; Chimonas 1970) have been shown to possess parallels in stability analyses of axisymmetric disturbances in an inviscid swirling flow. In particular, Howard & Gupta (1962) examined the stability of a swirling, incompressible, constant density, inviscid flow for which the pressure, axial velocity and azimuthal velocity are functions of the radius r only and showed that an equivalent 'Richardson' criterion can be stated for the stability of such flows against axisymmetric disturbances. They also noted that, if the density is constant and non-axisymmetric disturbances are present, no purely stabilizing term exists and consequently no stability condition can be obtained by their method. Leibovich (1969) extended the results of Howard & Gupta (1962), for axisymmetric disturbances only, to incompressible flows of variable density. Kurzweg (1969) also considered the same flow, but studied its stability against all disturbances. The resulting sufficient condition for stability was expressed in terms of two simultaneous inequalities that were fairly complicated functions of the basic flow quantities. No 'Richardson' number could be defined through the use of these inequalities. Recently, Howard (1973) returned to the problem and investigated the effect of compressibility. He considered axisymmetric disturbances only and showed that the results of Chimonas (1970) for the stability of compressible stably stratified shear flow possess an analogue in compressible swirling flow and that this can be expressed in terms of an effective 'Brunt–Väisälä' frequency. Because non-axisymmetric disturbances are known to be the most unstable ones (see Pedley 1968; Lessen, Sadler & Lin 1968, for example), it is important to investigate whether Howard's (1973) results can be extended to include general disturbances of arbitrary asymmetry.

The results of the present work can be summarized in the following theorem. *The stability of compressible non-dissipative swirling flow to all small disturbances is assured if*

$$N^2 - \tfrac{1}{4}[W'^2 + (V' - V/r)^2] \geqslant 0,$$

where $$N^2 = r\left(\frac{V}{r}+\sigma\right)^2\left[\frac{\rho_0'}{\rho_0} - r\left(\frac{V}{r}+\sigma\right)^2\Big/C_0^2\right],$$

ρ_0, C_0, V *and* W *are the density, sound speed, azimuthal velocity and axial velocity respectively, all being functions of the radius* r *only, and* σ *is the constant angular velocity of the frame of reference* (if any). A prime indicates differentiation with respect to r.

In addition to the stability criterion, an upper bound to the growth rate of small disturbances of arbitrary configuration is obtained. Finally, the semi-circle theorem (Howard 1961) is extended to azimuthal disturbances to incompressible, constant density, swirling flows, but further attempts to derive general necessary conditions for instability were unsuccessful.

2. Equations of motion

The equations governing the isentropic motion of a compressible fluid relative to a rotating frame of angular velocity $\boldsymbol{\sigma}$ in the presence of gravity are

$$D\rho/Dt + \rho\nabla.\,\mathbf{q} = 0, \tag{2.1}$$

$$\rho\left[\frac{D\mathbf{q}}{Dt} + 2\boldsymbol{\sigma}\times\mathbf{q} + \boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{r})\right] = -\nabla p + \rho\mathbf{g}, \tag{2.2}$$

$$Ds/Dt = 0, \quad s = s(p,\rho). \qquad (2.3), (2.4)$$

Here ρ is the density, p the pressure, s the specific entropy, $\mathbf{q}$ the velocity, $\mathbf{g}$ the gravitational acceleration and $\mathbf{r}$ the radius vector. The operator D/Dt is defined as usual by

$$D/Dt = \partial/\partial t + \mathbf{q}.\nabla. \tag{2.5}$$

Equation (2.4) is the equation of state for the fluid, and in conjunction with (2.3) gives

$$Dp/Dt - C^2 D\rho/Dt = 0, \tag{2.6}$$

where

$$C^2 \equiv [\partial p/\partial\rho]_s. \tag{2.7}$$

The system of equations is now comprised of (2.1), (2.2) and (2.6) along with the definition (2.7).

Let us introduce a cylindrical co-ordinate system (r,θ,z) rotating with a constant angular velocity $\boldsymbol{\sigma}$ parallel to the z axis and neglect gravity. Then (2.1), (2.2) and (2.6) are satisfied in the cylindrical region $r_1 \leqslant r \leqslant r_2$ by any flow with

$$\mathbf{q}(r) = V(r)\,\mathbf{e}_\theta + W(r)\,\mathbf{e}_z, \tag{2.8}$$

$$p_0(r) = \int_{r_1}^{r}\rho_0\left(\frac{V}{r}+\sigma\right)^2 r\,dr \tag{2.9}$$

and $\rho_0(r)$, $V(r)$ and $W(r)$ arbitrary given functions of r.

We are interested in examining the stability of the basic state (2.8), (2.9) to infinitesimal perturbations of the form

$$\begin{bmatrix} p_1(r,\theta,z;t) \\ \rho_1(r,\theta,z;t) \\ q_{1r}(r,\theta,z;t) \\ q_{1\theta}(r,\theta,z;t) \\ q_{1z}(r,\theta,z;t) \end{bmatrix} = \begin{bmatrix} \tilde{p}_1(r) \\ \tilde{\rho}_1(r) \\ v_r(r) \\ v_\theta(r) \\ v_z(r) \end{bmatrix} \exp\left[i(\omega t - m\theta - kz)\right], \tag{2.10}$$

where $\omega = \omega_r + i\omega_i$ ($\omega_i < 0$) is the complex frequency (whose imaginary part is assumed non-zero) with m and k the azimuthal and axial wavenumbers, both taken to be real.

Linearization of the basic equations (2.1), (2.2) and (2.6) about a basic flow given by (2.8) and (2.9) yields the following set of linearized equations for the perturbation quantities $\tilde{p}_1$, $\tilde{\rho}_1$, v_r, v_θ and v_z:

$$i\Omega\tilde{\rho}_1 + v_r\rho_0' + \rho_0\left[\frac{dv_r}{dr} + \frac{v_r}{r} - \frac{im}{r}v_\theta - ikv_z\right] = 0, \tag{2.11}$$

$$i\rho_0\Omega v_r - 2\rho_0\left(\frac{V}{r}+\sigma\right)v_\theta - \tilde{\rho}_1\left(\frac{V^2}{r} + 2\sigma V + \sigma^2 r\right) = -\frac{d\tilde{p}_1}{dr}, \tag{2.12}$$

$$i\rho_0\Omega v_\theta + \rho_0(V' + V/r + 2\sigma)v_r = (im/r)\tilde{p}_1, \tag{2.13}$$

$$i\rho_0\Omega v_z + \rho_0 W' v_r = ik\tilde{p}_1, \tag{2.14}$$

$$i\Omega\tilde{p}_1 + v_r p_0' = C_0^2(i\Omega\tilde{\rho}_1 + v_r\rho_0'). \tag{2.15}$$

In the above, $C_0^2(r) = [\partial p_0/\partial\rho_0]_s$ is the square of the sound speed and Ω is given by

$$\Omega = \omega - mV/r - kW. \tag{2.16}$$

Primes denote differentiation with respect to r.

Eliminating $\tilde{p}_1$, $\tilde{\rho}_1$, v_θ and v_z in favour of v_r from (2.11)–(2.15), we obtain

$$\begin{aligned} &\frac{d}{dr}\left\{\rho_0\frac{\Omega S}{\Delta}\left[\frac{dv_r}{dr} + \frac{v_r}{r} + v_r\left(\frac{m}{r\Omega}\left(V' + \frac{V}{r} + 2\sigma\right) + \frac{kW'}{\Omega} + \frac{p_0'}{\rho_0 C_0^2}\right)\right]\right\} \\ &\quad - \left[\frac{1}{C_0^2}\frac{(V+r\sigma)^2}{r} + \frac{2m}{r\Omega}\left(\frac{V}{r}+\sigma\right)\right]\left\{\rho_0\frac{\Omega S}{\Delta}\left[\frac{dv_r}{dr} + \frac{v_r}{r}\right.\right. \\ &\quad \left.\left. + v_r\left(\frac{m}{r\Omega}\left(V' + \frac{V}{r} + 2\sigma\right) + \frac{kW'}{\Omega} + \frac{p_0'}{\rho_0 C_0^2}\right)\right]\right\} - \rho_0\Omega v_r\left[1 - \frac{N^2}{\Omega^2} - \frac{\Phi}{\Omega^2}\right] = 0, \end{aligned} \tag{2.17}$$

where $$S = r^2/(m^2 + r^2k^2), \quad \Delta = 1 - S\Omega^2/C_0^2, \qquad (2.18), (2.19)$$

$$N^2 = \left(\frac{\rho_0'}{\rho_0} - \frac{p_0'}{\rho_0 C_0^2}\right)\frac{(V+\sigma r)^2}{r}, \tag{2.20}$$

$$\Phi = 2(V/r + \sigma)(V' + V/r + 2\sigma). \tag{2.21}$$

N is the effective Brunt–Väisälä frequency and here is assumed positive to assure static stability; Φ is the Rayleigh discriminant.

Let us define

$$E(r) = \exp\left[-\int^r \left(\frac{p_0'}{\rho_0 C_0^2}\right) dr\right], \tag{2.22}$$

$$q(r) = (v_r(r)/i\Omega)\, E(r) \tag{2.23}$$

and

$$R = \rho_0/E^2. \tag{2.24}$$

With these definitions (2.17) becomes

$$\left\{\frac{RS\Omega}{\Delta}\left[\Omega\left(q'+\frac{q}{r}\right)+\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)q\right]\right\}' - \frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\frac{RS}{\Delta}\left[\Omega\left(q'+\frac{q}{r}\right)+\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)q\right]$$
$$-R[\Omega^2 - N^2 - \Phi]\,q = 0. \tag{2.25}$$

If the fluid is contained in an annular cylinder with rigid walls at r_1, which may be zero, and $r_2 > r_1$, the boundary conditions to be imposed on v_r and hence on q are that they must vanish at r_1 and r_2. For the special case when $|m| = 1$ and $r_1 = 0$, the boundary condition $v_r = 0$ at $r = r_1 = 0$ has to be replaced by the condition $v_r' = 0$.

3. The sufficient condition for stability

Following Howard (1973), let us assume that the flow is unstable, so that ω_i, the imaginary part of the frequency, is negative definite, and define a new variable

$$\phi = q\Omega^{\frac{1}{2}}. \tag{3.1}$$

Substituting (3.1) into (2.25) we obtain

$$\left[\frac{R\Omega S}{\Delta}\left(\phi'+\frac{\phi}{r}\right)\right]' - \frac{\Omega'^2}{4\Omega^2}\frac{RS}{\Delta}\phi + \frac{\Omega'}{2}\frac{RS}{\Delta}\left[1+\frac{4m}{\Omega}\left(\frac{V}{r}+\sigma\right)\right]\frac{\phi}{r}$$
$$-\left\{\frac{SR}{2\Delta}\left[\Omega' - \frac{4m}{r}\left(\frac{V}{r}+\sigma\right)\right]\right\}'\phi - \frac{RS}{\Delta}\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\left[1+\frac{2m}{\Omega}\left(\frac{V}{r}+\sigma\right)\right]\frac{\phi}{r}$$
$$-R\Omega\left[1-\frac{N^2}{\Omega^2}-\frac{\Phi}{\Omega^2}\right]\phi = 0. \tag{3.2}$$

Multiplication of (3.2) by $r\phi^*$, where ϕ^* is the complex conjugate of ϕ, integration of the resulting equation between r_1 and r_2, and use of the boundary conditions results in

$$\int_{r_1}^{r_2} dr\left\{-\frac{R\Omega S}{\Delta}\left[r|\phi'|^2 + (\phi\phi^*)' + \frac{|\phi|^2}{r}\right] - \frac{\Omega'^2}{4\Omega}\frac{RS}{\Delta}r|\phi|^2\right.$$
$$+\frac{\Omega'}{2}\frac{RS}{\Delta}\left[2|\phi|^2 + r(\phi\phi^*)' + \frac{4m}{\Omega}\left(\frac{V}{r}+\sigma\right)|\phi|^2\right]$$
$$-\frac{RS}{\Delta}\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\left[|\phi|^2 + r(\phi\phi^*)' + \frac{2m}{\Omega}\left(\frac{V}{r}+\sigma\right)|\phi|^2\right]$$
$$\left. -rR\left[\Omega - \frac{N^2}{\Omega} - \frac{\Phi}{\Omega}\right]|\phi|^2\right\} = 0. \tag{3.3}$$

The imaginary part of (3.3) is

$$\omega_i\left\{\int_{r_1}^{r_2}\frac{R\,dr}{|\Delta|^2}\left[\frac{S}{r}\left(1+\frac{|\Omega|^2S}{C_0^2}\right)|(r\phi)'|^2-2\frac{\Omega_r S^2}{C_0^2}\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right][\phi^*(r\phi)'+\phi(r\phi^*)']\right.\right.$$
$$\left.+\frac{S^2}{C_0^2}\left(1+\frac{|\Omega|^2S}{C_0^2}\right)\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2 r|\phi|^2\right]+\int_{r_1}^{r_2}dr\,rR|\phi|^2$$
$$\left.+\int_{r_1}^{r_2}r\,dr\frac{R}{|\Omega|^2}\left[(N^2+\Phi)-S\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2\right]|\phi|^2\right\}=0, \quad (3.4)$$

where Ω_r is the real part of Ω.

The integrand I_0 of the first integral in (3.4) can be rewritten as the sum of two terms in the following manner:

$$I_0=\frac{RS}{r|\Delta|^2}\left(1+\frac{|\Omega|^2S}{C_0^2}\right)(I_1+I_2),$$

with

$$I_1=|(r\phi)'|^2+\frac{S}{C_0^2}\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2 r^2|\phi|^2$$
$$-2\left(\frac{S}{C_0^2}\right)^{\frac{1}{2}}\left|\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right| r|\phi|\,|(r\phi)'|$$

and

$$I_2=2\left(\frac{S}{C_0^2}\right)^{\frac{1}{2}}\left|\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right| r|\phi|\,|(r\phi)'|$$
$$-r\frac{S}{C_0^2}\frac{2\Omega_r}{(1+|\Omega|^2S/C_0^2)}\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right][\phi^*(r\phi)'+\phi(r\phi^*)'].$$

The first term I_1 is a perfect square. The second term I_2 is also non-negative since

$$2|\phi|\,|(r\phi)'|\geqslant\phi^*(r\phi)'+\phi(r\phi^*)'$$

and

$$\left(\frac{S}{C_0^2}\right)^{\frac{1}{2}}\geqslant\frac{2\Omega_r S/C_0^2}{1+|\Omega|^2S/C_0^2}.$$

It follows that I_0 is non-negative and the first integral in (3.4) is non-negative. Equation (3.4) can be written as

$$\omega_i\left\{\int_{r_1}^{r_2}I_0\,dr+\int_{r_1}^{r_2}rR|\phi|^2\,dr\right.$$
$$\left.+\int_{r_1}^{r_2}r\,dr\frac{R}{|\Omega|^2}\left\{(N^2+\Phi)-S\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2\right\}|\phi|^2\right\}=0. \quad (3.5)$$

The second integral in (3.5) is positive. If

$$N^2+\Phi-\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2\Big/\left(\frac{m^2}{r^2}+k^2\right)\geqslant 0 \quad (3.6)$$

throughout the range $r_1\leqslant r\leqslant r_2$, the third integral in (3.5) will also be non-negative and (3.5) will not be satisfied unless $\omega_i=0$. Thus a sufficient condition for stability is that (3.6) holds throughout the domain of the flow.

Following Howard (1961, §4) we obtain from (3.5) the following bound for the growth rate ω_i:

$$\omega_i^2 \leqslant \max\left(\frac{1}{4}\left\{\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2\Big/\left(\frac{m^2}{r^2}+k^2\right)\right\}-N^2-\Phi\right), \tag{3.7}$$

where the maximum is over the domain $[r_1, r_2]$.

In an effort to obtain a criterion that does not depend explicitly on the wavenumbers, we note that

$$\left[\frac{\Omega'}{2}-\frac{2m}{r}\left(\frac{V}{r}+\sigma\right)\right]^2\Big/\left(\frac{m^2}{r^2}+k^2\right) \leqslant \frac{1}{4}\left[W'^2+\left(V'+\frac{3V}{r}+4\sigma\right)^2\right], \tag{3.8}$$

and so if

$$N^2+\Phi-\tfrac{1}{4}[W'^2+(V'+3V/r+4\sigma)^2] \geqslant 0 \tag{3.9}$$

then (3.6) is also satisfied. Rearranging (3.9) we obtain

$$N^2-\tfrac{1}{4}[W'^2+(V'-V/r)^2] \geqslant 0 \tag{3.10}$$

for stability. This proves the theorem stated in the introduction. The effective 'Richardson number' that results is $N^2/[W'^2+(V'-V/r)^2]$. With the help of (3.8), the expression for the growth rate (3.7) becomes

$$\omega_i^2 \leqslant \max\{\tfrac{1}{4}[W'^2+(V'-V/r)^2]-N^2\}, \tag{3.11}$$

which does not involve the wavenumbers explicitly but is not as stringent as (3.7).

4. The semicircle theorem for azimuthal disturbances

While this paper was in preparation, Maslowe (1974) presented some new results on the stability of incompressible rigidly rotating flows. He proved that for such flows a necessary condition for instability is that

$$\Omega_r = \omega_r - mV/r - kW$$

be zero somewhere in $r_1 \leqslant r \leqslant r_2$. Furthermore, he calculated numerically the growth rates for non-axisymmetric disturbances with azimuthal wavenumbers of -1 and -4. An attempt to extend Maslowe's necessary condition for instability to heterogeneous flows with non-axisymmetric disturbances was unsuccessful. We were able, though, to demonstrate that the 'semicircle' theorem (Howard 1961) holds for azimuthal disturbances in incompressible flows with arbitrary axial and swirl velocity components. To show this, we return to (2.25), multiply it by rq^* and integrate from r_1 to r_2. The result is

$$\int_{r_1}^{r_2} dr\left\{\frac{R\Omega^2 S}{\Delta}\frac{1}{r}|(rq)'|^2+\frac{R\Omega S}{\Delta}\frac{2m}{r^2}\left(\frac{V}{r}+\sigma\right)(r^2|q|^2)' + \frac{RS}{\Delta}4m^2\left(\frac{V}{r}+\sigma\right)^2\frac{|q|^2}{r}+R[\Omega^2-N^2-\Phi]r|q|^2\right\} = 0. \tag{4.1}$$

When $m = 0$, the above equation yields (Howard 1973) the semicircle theorem for axisymmetric disturbances.

If we now restrict our attention to incompressible flows and azimuthal disturbances, so that $C_0^{-2} = 0$ and $k = 0$, we obtain from the real and imaginary parts, respectively, of (4.1)

$$\int_{r_1}^{r_2} \rho_0 dr \left\{ (\Omega_r^2 - \omega_i^2)\, r|(rq)'|^2 + 2m\left(\frac{V}{r}+\sigma\right)\Omega_r\,(r^2|q|^2)' + 4m^2\left(\frac{V}{r}+\sigma\right)^2 r|q|^2 + m^2[(\Omega_r^2-\omega_i^2) - N^2 - \Phi]\, r|q|^2 \right\} = 0 \qquad (4.2)$$

and

$$2\omega_i \int_{r_1}^{r_2} \rho_0 dr \left\{ \Omega_r r|(rq)'|^2 + m\left(\frac{V}{r}+\sigma\right)(r^2|q|^2)' + m^2\Omega_r r|q|^2 \right\} = 0. \qquad (4.3)$$

In (4.2) and (4.3) Ω_r is reduced to

$$\Omega_r = \omega_r - mV/r.$$

Equation (4.3) can be rewritten as

$$\omega_i \int_{r_1}^{r_2} \rho_0 dr \Omega_r [r|(rq)'|^2 - (r^2|q|^2)' + m^2 r|q|^2] = -\omega_i \int_{r_1}^{r_2} (\omega_r + m\sigma)\rho_0 dr (r^2|q|^2)'. \qquad (4.4)$$

When ρ_0 = constant, the right-hand side of (4.4) is zero, because of the boundary conditions. Furthermore, the square bracket on the left-hand side,

$$P_1 = r|(rq)'|^2 - (r^2|q|^2)' + m^2 r|q|^2,$$

can be expanded so that it takes the form

$$P_1 = r^3|q'|^2 + r^2(qq^*)' + r|q|^2 - r^2(qq^*)' - 2r|q|^2 + m^2 r|q|^2$$
$$= r^3|q'|^2 + (m^2-1)\, r|q|^2 \geqslant 0,$$

since $|m| \geqslant 1$. Then (4.4) can be satisfied only if either $\omega_i = 0$ or $\Omega_r = \omega_r - mV/r$ is zero somewhere in $r_1 \leqslant r \leqslant v_2$, and so a necessary condition for instability of azimuthal disturbances is that ω_r/m be equal to V/r somewhere in $r_1 \leqslant r \leqslant r_2$.

Upon rearrangement, (4.2) and (4.4) become

$$\int_{r_1}^{r_2} \rho_0 dr (\Omega_r^2 - \omega_i^2) P_1 = 0 \qquad (4.5)$$

and

$$\int_{r_1}^{r_2} \rho_0 dr \Omega_r P_1 = 0. \qquad (4.6)$$

From (4.5) and (4.6), following Howard (1961, §3) the semicircle theorem is obtained, i.e. ω_r and ω_i are restricted such that

$$[\omega_r/m - \tfrac{1}{2}(a+b)]^2 + (\omega_i/m)^2 \leqslant [\tfrac{1}{2}(a-b)]^2 \qquad (4.7)$$

holds. In (4.7), a and b are constants such that $a \leqslant V/r \leqslant b$.

This research was supported in part by Grant GA-40243 from the National Science Foundation.

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